PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON TUESDAY, FEBRUARY 27, 2024, on Sec. 10.1: Curves Defined by Parametric Equations and Sec. 10.2: CALCULUS WITH PARAMETRIC EQUATIONS

CLASS #13

View the Conic Sections Videos: Parabolas, Ellipses, Hyperbolas

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Describing Curves with Parametric Equations

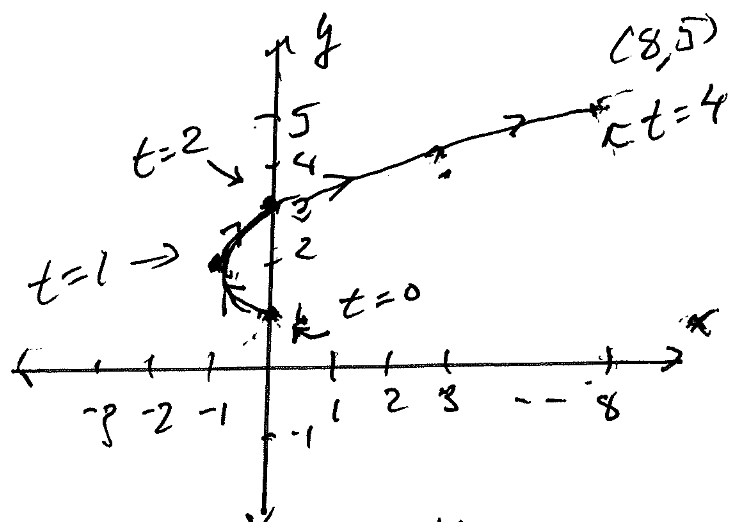
EXAMPLE: Sketch the curve given by the Equations

$x = f(t) = t^2 - 2t$

$y = g(t) = t + 1$

where $0 \le t \le 4$

t	Point (x, y): $(t^2 - 2t, t+1)$
0	$(0, 1)$
1	$(-1, 2)$
2	$(0, 3)$
3	$(3, 4)$
4	$(8, 5)$



The arrows indicate the "Direction of increasing Values of the parameter."

P. 666

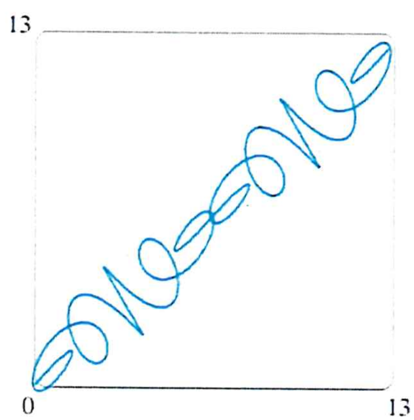


FIGURE 12
$x = t + \sin 5t$
$y = t + \sin 6t$

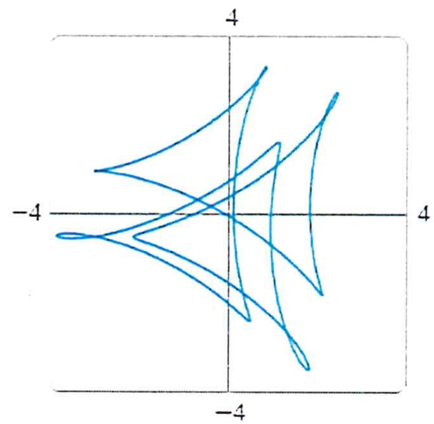


FIGURE 13
$x = \cos t + \cos 6t + 2 \sin 3t$
$y = \sin t + \sin 6t + 2 \cos 3t$

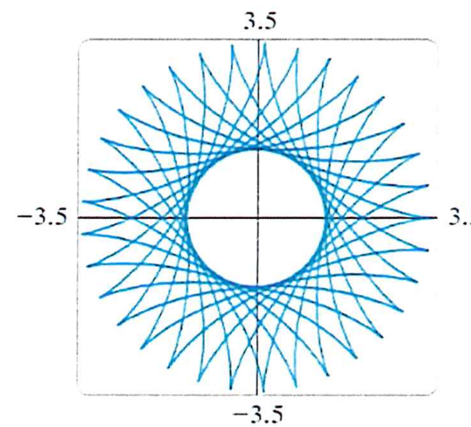


FIGURE 14
$x = 2.3 \cos 10t + \cos 23t$
$y = 2.3 \sin 10t - \sin 23t$

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Eliminating the Parameter is finding an xy equation that has, as its graph, the same curve described by the parametric equations.

Example:

Consider the Curve

$x = t-1$

$y = 2t^2 - 4t + 1$

$0 \le t \le 2$

Method (1)

Solve for t from one of the equations in terms of x (or of y, or both). Then, substitute the expression that calculates t into the other equation, and everywhere (including in the restrictions on t).

From $x = t - 1$,

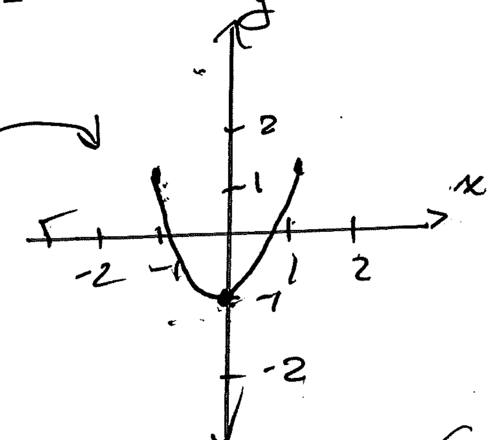
$t = x + 1$.

Substitute $t = x+1$ into $y = 2t^2 - 4t + 1$, giving

$y = 2(x+1)^2 - 4(x+1) + 1$

and $0 \le x+1 \le 2 \Rightarrow -1 \le x \le 1$

$$\boxed{y = 2x^2 - 1 \text{ and } -1 \le x \le 1 \text{ and } -1 \le y \le 1}$$



EXAMPLE:

Consider this Curve:

$x = 2\cos t$

$y = 2\sin t$

$0 \le t \le 2\pi$.

Sol'n $\cos^2 t + \sin^2 t = 1$

From $x = 2\cos t$,

$\cos t = \frac{x}{2}$

From $y = 2\sin t$,

$\sin t = \frac{y}{2}$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

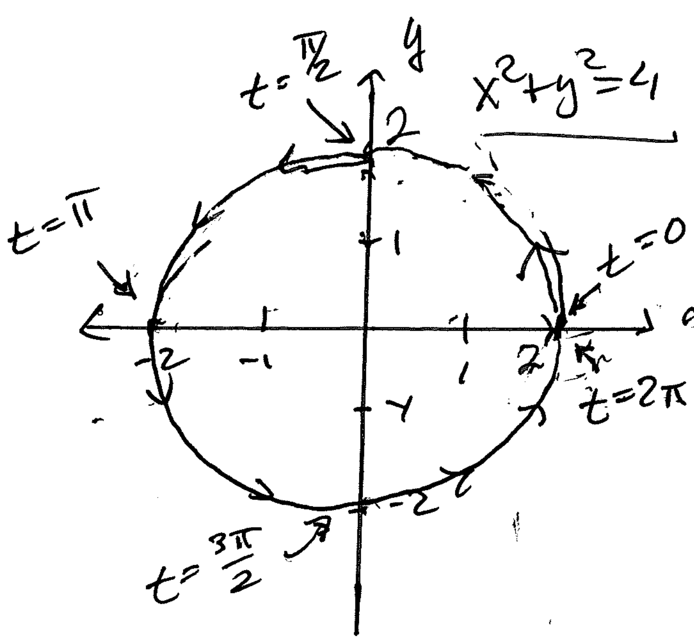
$\hookrightarrow \frac{x^2}{4} + \frac{y^2}{4} = 1$

$\hookrightarrow x^2 + y^2 = 4$

A circle of Radius $r = 2$ with Center $(0,0)$

method ②:

Find a relationship between the functions involved in the equations and solve for them in terms of x and y, then use the relationship to make an equation in x and y.



EXAMPLE:

$$\left.\begin{array}{l} x = 3\cos t \\ y = 2\sin t \end{array}\right\} \Rightarrow \begin{array}{l} \cos t = \frac{x}{3} \\ \sin t = y/2 \end{array}$$

$\cos^2 t + \sin^2 t = 1$

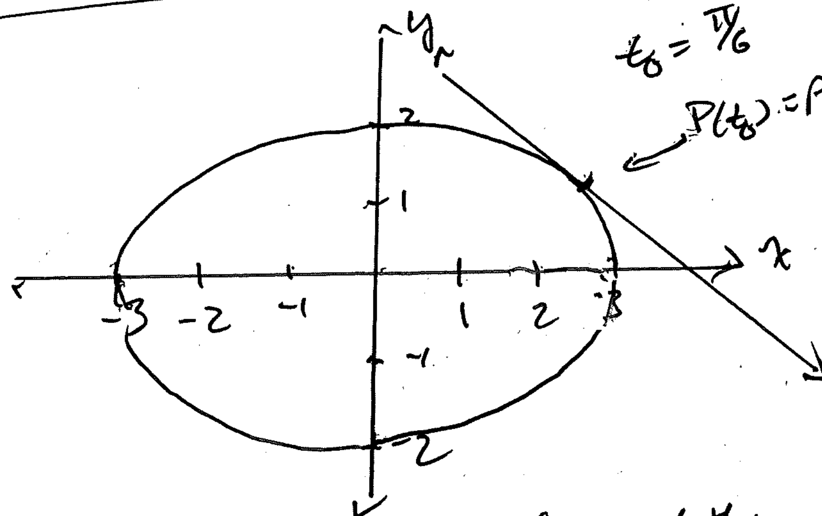
$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

An ellipse

$-3 \le x \le 3$

$-2 \le y \le 2$



What is the slope of the the tangent line at $P(t_0)$ with $t_0 = \pi/6$?

FACT: For the curve described by $x = f(t)$ and $y = g(t)$, the slope of the curve at the point for t is

$$\text{slope} = \frac{dy/dt}{dx/dt}$$

In the example above; $x = 3\cos t$ $y = 2\sin t$

$$\left.\begin{array}{l} \frac{dx}{dt} = -3\sin t \\ \frac{dy}{dt} = 2\cos t \end{array}\right\} \quad \text{slope} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\text{Slope} = \frac{2\cos t}{-3\sin t} \text{ at the point } P(t)$$

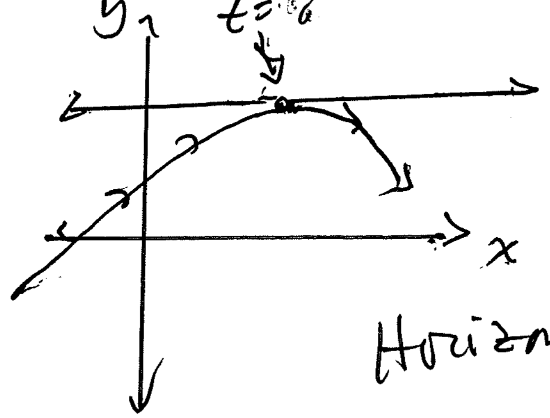
slope of tangent line at $t = t_0 = \pi/6$ is $\frac{2\cos(\pi/6)}{-3\sin(\pi/6)} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{-3\left(\frac{1}{2}\right)} = \frac{\sqrt{3}}{\left(-\frac{3}{2}\right)} = -\frac{2}{3}\sqrt{3}$

≈ -1.155

Horizontal and Vertical Tangent Lines

Param
$x = f(t)$
$y = g(t)$

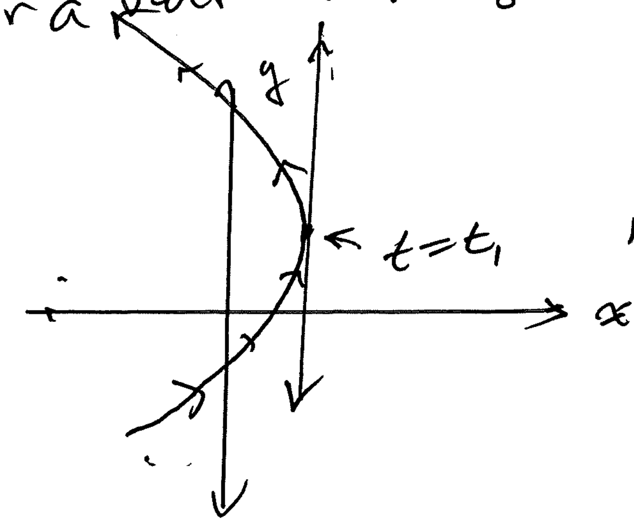
For a Horizontal Tangent Line:



$$\frac{dy}{dt} = 0 \text{ and } \frac{dx}{dt} \neq 0.$$

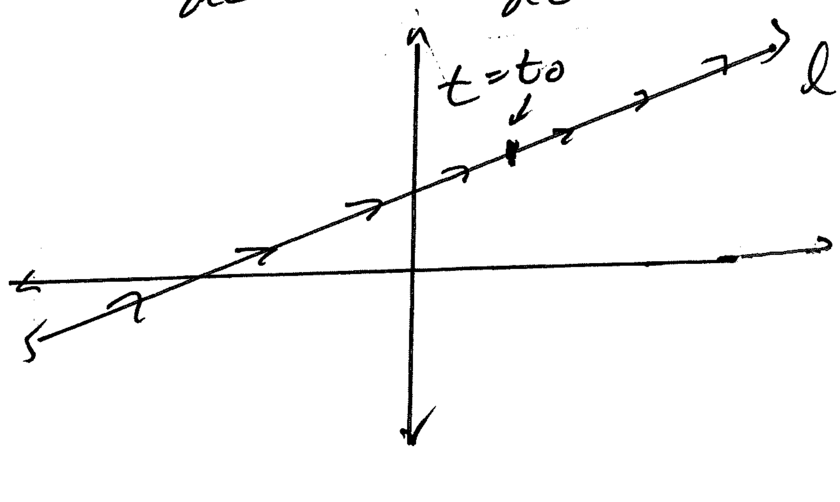
Horizontal Tangent at $P(t_0)$
$t = t_0$

For a Vertical Tangent: $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

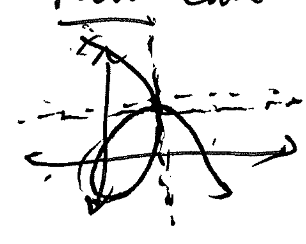


A. Vertical Tangent at $t = t_1$

If both $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, Anything might be happening



Be aware this can happen:



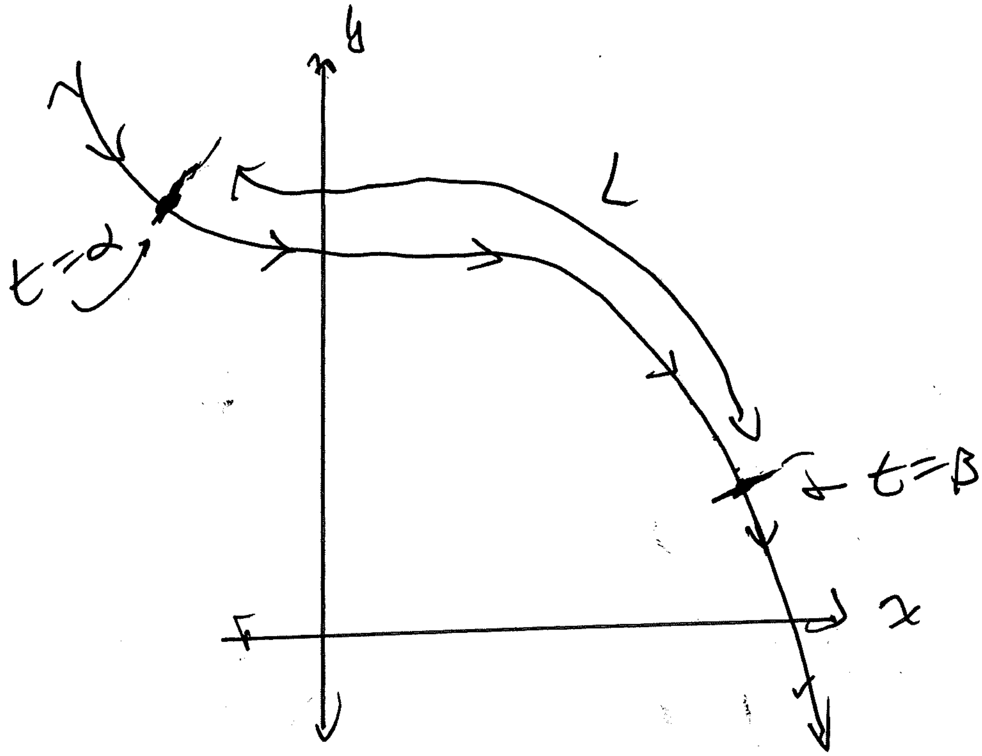
The Arclength L of a Segment of a Curve

For the Curve: $x = f(t)$
$y = g(t)$

and, for the part of the curve corresponding to $\alpha \le t \le \beta$,

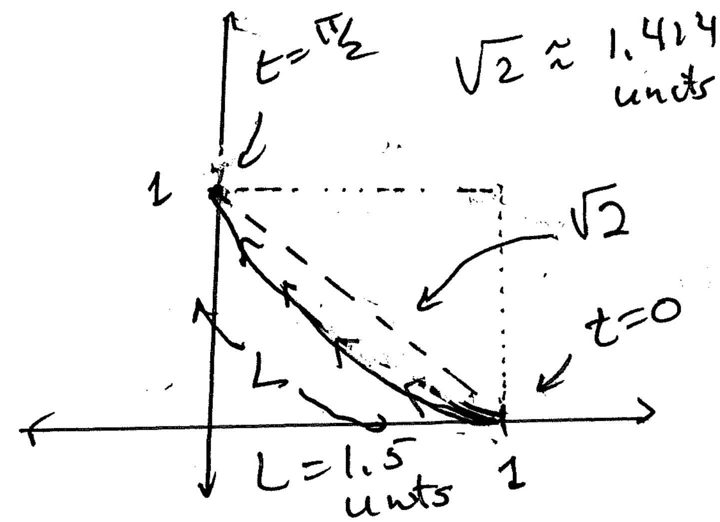
The length L of arc (when this part of the curve is traversed exactly once) is given by

$$\text{arclength } L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



Ex: Find the length L of the arc of the curve

$$\left.\begin{array}{l} x = \cos^3 t \\ y = \sin^3 t \end{array}\right\} \text{for } \alpha = 0 \text{ and } \beta = \pi/2$$



Sol'n

$$\frac{dx}{dt} = 3\cos^2(t)(-\sin t)$$

$$\frac{dy}{dt} = 3\sin^2(t)(\cos t)$$

$$L = \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)}\, dt$$

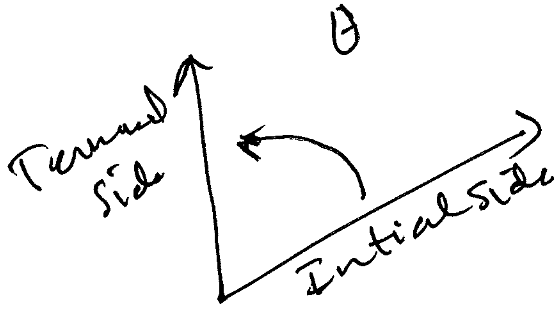
$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t}\, dt = \int_0^{\pi/2} \sqrt{(3\cos t \sin t)^2}\, dt$$

$$= \int_0^{\pi/2} |3\cos t \sin t|\, dt = \int_0^{\pi/2} 3\cos t \sin t\, dt$$

$$= \dots = \frac{3}{2} \text{ units} = 1.5 \text{ units}$$

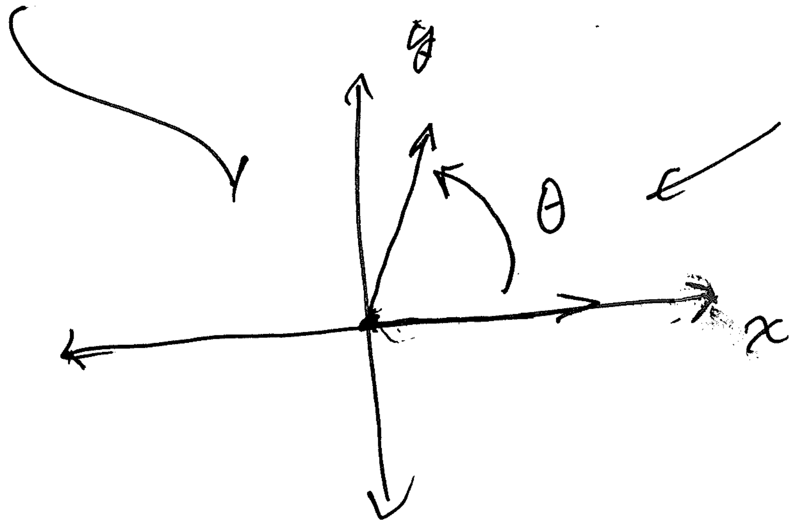
Angles θ in Trig

are Angles θ of Rotation.



If $\theta > 0$, The Rotation is counter-clockwise.

If $\theta < 0$, The Rotation is clockwise



Angle θ in Standard Position